REVIEW

on the PhD thesis of Sabitbek Bolys Mazhituly «**Hardy-Sobolev type inequalities on homogeneous groups and applications**», submitted for the degree of Doctor of Philosophy (PhD) in the speciality "6D060100 - Mathematics"

1. The relevance of the research topic and its relationship with general scientific and national programs.

The study of the subelliptic functional estimates has begun more than couple decades ago due to their importance for many questions involving subelliptic partial differential equations, unique continuation, sub-Riemannian geometry, subelliptic spectral theory, etc. In 1990, Garofalo and Lanconelli firstly established the subelliptic Hardy inequality on the Heisenberg group which is the most important example of the homogeneous groups.

In recent years, the subelliptic functional inequalities and related analysis on the homogeneous groups have been a topic of intensive research summarised in the very recent appearing book titled *Hardy inequalities on homogeneous groups* by Ruzhansky and Suragan. This book covers the most recent developments of the subelliptic functional inequalities on the Heisenberg groups, the stratified Lie groups, the graded Lie groups, and the homogeneous groups.

Note that, the most works in the PhD thesis are included in that book «Hardy inequalities on homogeneous groups», and B. Sabitbek is listed as the contributors in this book.

2. Scientific results and their validity.

The PhD thesis includes several important results for the subelliptic analysis such as:

1. There are established the geometric Hardy inequality on the half-spaces of stratified groups which gives the proof of the conjecture by Simon Larson (Theorem 2.5.1).

2. Geometric Hardy-Sobolev inequality on the half-spaces of stratified groups is proved for the first time on the Heisenberg group.

3. The embedding theorem and Sobolev type spaces are consequences of the Hardy-Rellich type inequalities (Theorem 3.5.1).

4. Moreover, the weighted Hardy and Rellich inequalities for general vector fields without a group structure are proved, which recover most of the previous results.

3. The degree of validity and reliability of each scientific result, proofs and conclusions of the applicant formulated in the thesis.

All the obtained results are fully scientifically substantiated and sufficiently correct: they are supplied with accurate and complete mathematical calculations. All evidence presented is correct. The results obtained by the applicant have been presented to the international conferences and scientific seminars. The conclusions of the mathematics follow exactly from the provisions submitted for defense.

4. The degree of novelty of each scientific result, proof of the applicant formulated in the PhD thesis.

In the PhD thesis, all established results are completely new.

There are enough results that have been published in a peer-reviewd scientific journals by the Web of Science and Scopus with a non-zero impact factor: Complex Analysis and Operator Theory, Mathematical Modelling and Natural Phenomena, Revista Matematica Complutense, Nonlinear Differential Equations and Applications.

5. Practical and theoretical signficance of scientific results.

The results obtained in the PhD thesis is the fundamental reseach and can be applied to the subelliptic PDE and subelliptic spectral theory.

6. Comments, suggestions for the PhD thesis.

I suggest the author of the PhD thesis to take into account the following very minor inaccuracies in English and typos that need to be corrected:

- In the page 7, line 12: "On green functions for Dirichelet..." should be "On Green functions for Dirichlet...".
- In the page 10, first line after Definition 1.3.1 "Here, we say that r is called a step of G..." probably should be "Here, we say that r is a step of G..." or "Here, r is called a step of G... ".
- In the page 14, line 7 in the last paragraph: there is missed Hq, and consequently, should be "The Haar measure on Hq coincides...".
- In the page 33, line 2: The word "respectively" is unnecessary.
- In the page 37, Theorem 2.6.2: Q is probably equal to 2n+2, not to 2n+1.
- In the page 61, first line after Definition 3.5.1: Instead of "Sovolev" should be "Sobolev".
- In the page 78,line 5: Instead of "...the left-hand side of (23),..." should be "...the left-hand side of (4.23),...".
- In Conclusion, line 4: Instead of "...the establishedfb result..." should be "...the established result...".

These comments do not affect the positive assessment of the work as a whole.

7. Compliance with the content of the PhD thesis in the framework of the requirements of the Rules for the award of scientific degrees.

The PhD thesis was performed at a high scientific level; it is a complete scientific work. This PhD thesis is clearly and correctly written, well related with the current literature on its subject. The content of the PhD thesis on "Hardy-Sobolev type inequalities on homogeneous groups and applications" meets all the requirements for the thesis for the degree of Doctor of Philosophy (PhD), and its author, Bolys Mathituly Sabitbek, deserves the award of the degree of Doctor of Philosophy (PhD) on specialty «6D060100-Mathematics».

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